On Some Physical Applications of Hamilton's Operator

by Frederick William Owens

1902

Submitted to the Department of Mathematics of the University of Kansas in partial fulfillment of the requirements for the Degree of Master of Science

Master thesis

Mathematics

Owens, F. W. 1902

On some physical applications of Hamilton's operator.

Passed "One".

"On Some Physical Applications of Hamilton's Operator ∇."

Frederick William Owens.

Thesis for Master's Degree. 1902.

On Some Physical Applications of Hamilton's Operator ∇.

It is the purpose of this paper to give simply an exposition of certain quaternion methods of investigation, involving the operator ∇. Certain of the formulae and theorems of mathematical physics are derived by quaternion methods, and an attempt is made to show that the same results are obtained, and in many cases, by less difficult processes than by ordinary analytical methods. By a proper choice of conditions, the various simple quaternion relations and transformations can be made to give a great number of these physical formulae, previously deduced, it is true, by analytical methods, but often with considerable difficulty.

The secret of the simplicity of

quaternion methods lies largely in the absolute freedom of treatment of direction in space. In analytical methods, we can treat of directed quantities only by taking their components along the axes of reference, a laborious method, at best, while by the methods of quaternion analysis, direction in space is as easily handled, and vector quantities and expressions are as easily interpreted as are the scalar magnitudes. In distinguishing the kinds of physical quantities, it is of great importance to know how they are related to the directions of the coordinate axes, which are usually employed in defining the position of things. By giving the quaternion expressions the forms involving i, j, k, these are easily obtained.

The introduction of coordinate axes into geometry by Des Cartes was one of the greatest steps in the progress of mathematical analysis. The position of a point in space is made to depend on the lengths of three lines drawn in definite directions, usually at right angles to each other. Hence it reduced the methods of geometry to calculations performed on numerical quantities.

But for many purposes in physical reasoning, it is desirable to avoid explicitly introducing the cumbersome Cartesian coordinates, and to think directly of the position of a point in space, rather than on its three coordinates, and of the magnitude and direction of a force, rather than of its three components. This method of contemplating geometrical and physical quantities is more primitive and natural than the other, but it was little used until Hamilton invented his "Calculus of Quaternions".

Of this method, I can do no better than to quote Clerk Maxwell[1], the greatest of all mathematical physicists. "I am convinced, however, that the introduction of the ideas, as distinguished from the operations and methods of quaternions will be of great use to us in the study of all parts of our subject, and especially, in electrodynamics, where we have to deal with a number of physical quantities, the relations of which to each other can be expressed far more simply by a few words of Hamilton's than by the ordinary equations."

There are physical quantities, however, which are related to directions in space, which are not vectors. Quantities of this kind require nine numerical specifications, in order to completely deter-

1. Clerk Maxwell. "Electricity and Magnetism" vol. 1. p. 9.

mine them. Stresses and strains are examples of these quantities. They can be expressed in quaternion notation as linear and vector functions of a vector. In quaternions, the position of a point in space is defined by its vector from the origin. If at that point of space we have to consider some physical quantity whose value depends on the position of the point, that quantity is treated as a function of the vector drawn from the origin. The density of a body, its temperature, its hydrostatic pressure, and its potential, may be expressed as scalar functions of this vector. The resultant force at the point, the velocity of a fluid at the point, the velocity of rotation of an element of the fluid, etc., may be represented as vector functions of the vector.

If we take the integral along a line of the product of an element of the line, and the resolved part of the force, along that element, the line integral of the force thus found represents the work done on a body carried along that line. When the value of this integral depends only on the position of the ends of the line, it is called the potential. The geometrical nature of the relation between the force and the resultant potential was greatly elucidated by Hamilton. If V is the potential and $F = iX + jY + kZ$ is the force, then, in quaternion notation,

$$F = -\left(i\frac{\delta V}{\delta x} + j\frac{\delta V}{\delta y} + k\frac{\delta V}{\delta z}\right)$$

$$= -\left(i\frac{\delta V}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z}\right)V \qquad (1)$$

The operator within the parenthesis is represented by the symbol ∇. Then we have

$$F = \nabla V \qquad (2)$$

Then we may consider the symbol

of operation ∇ as directing us to measure in each of the directions, i, j, k, the rate of increase of V, and find their vector sum.

M. Lamé used[1] the term "First Differential Parameter" to express the magnitude of F, but he does not seem to have considered it as a vector quantity. Webster calls[2] it the "Vector Differential Parameter" and recognizes its character as a vector. Maxwell also saw its true character, and termed[3] F the "Slope" of the scalar function V.

The operation indicated by ∇, then may be seen from the above definition to be that operation by which a vector quantity may be deduced

1. "Leçons Sur les Coordonnées Curvilignes et Leur Diverses Applications" p. 6.
2. "Theory of Electricity and Magnetism." p. 22.
3. "Electricity and Magnetism." p 15.

from its potential. Hamilton, himself, defined[1] ∇, in the language of pure mathematics as an operator which produces a vector from a scalar function of a vector,

$$\nu = \nabla (f\rho). \qquad (3)$$

As a matter of convenience, the operand is placed to the right of the operator ∇ wherever possible, and where this is not done the operand is indicated by a subscript to the operator. If f and g are any two scalar functions of ρ then, since $d(f+g) = df + dg$,

$$\nabla(f+g) = \nabla f + \nabla g, \qquad (4a)$$

and since $d(fg) = g\,df + f\,dg$,

$$\nabla(fg) = g\nabla f + f\nabla g. \qquad (4b)$$

$$f\rho = C \qquad (5)$$

is the vector equation of a surface. Then the differential equation of the series of surfaces obtained by giving

1. "Elements of Quaternions" vol II p. 432.

arbitrary values to C may be written,

$d(f\rho) = 0$, or

$$d(f\rho) = S\nu\, d\rho = 0 \qquad (6)$$

Since $d\rho$ is any tangent vector, and two lines are at right angles when the scalar of their product is equal to zero, ν represents a vector perpendicular to the surface. If we take another surface near the surface just given, for which, $S\nu\delta\rho = \delta C$, then

$$f(\rho + \nu^{-1}\delta C) = C + \delta C \qquad (7)$$

Hence the length of ν is the normal distance between two consecutive surfaces.

The equation $df(\rho) = 0$ may be taken to represent an equipotential or isothermal surface, since the form is perfectly general. Then we see that $-\nu$ represents in direction as well as in magnitude the force, or the flux of heat at the point whose vector is ρ.

If we take the general differential equation of the surface, from (6)

$S\,\nu\,d\rho = 0$,
let us find the criterion of its integrability. We may assume that the integral, if one exists, is of the form $f\rho = C$. Applying the operator ∇,

$$\nabla f = \left(i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z}\right)f. \quad \text{Also,}$$

$$df = 0 = \left(\frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy + \frac{\delta f}{\delta z}dz\right) = -S\,d\rho\nabla f. \quad (7)$$

Comparing (6) and (7) we see that they are identical if ν, or some scalar multiple of it can be expressed as ∇f. If $\nu = \nabla f$,

$$\nabla\nu = \nabla^2 f = \left(i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z}\right)^2 f$$

Squaring the term in parenthesis,

$$\nabla\nu = \nabla^2 f = -\left(\frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} + \frac{\delta^2 f}{\delta z^2}\right) \qquad (8)$$

The last written quantity is the form well known in mathematical physics as Laplace's operator, and is a scalar quantity.

$\therefore \; V\nabla\nu = 0$ (9)

is the requisite condition of integrability. If ν does not satisfy this condition, there is still the chance that some scalar multiple of it will; i.e., $w\nu$, where w is a scalar, may do so. The condition may then be written, $V\nabla(w\nu) = 0$, or what is equivalent, $V\nu\nabla w - wV\nabla\nu = 0$, which requires that

$S\nu\nabla\nu = 0$ (10)

(9) and (10) together, then, constitute the desired condition of integrability.

From the equation $\nu = \nabla f\rho$, it follows that the effect of the vector operation indicated by ∇ upon any scalar function of the vector of a point is to produce the vector which represents in magnitude and direction the most rapid change in the value of the function.

From the definitions $\nu = \nabla f\rho$. Hence ∇ must be regarded as a symbolic vector; or at least, it must possess certain characteristics of a vector, since it produces a vector from a scalar. That it is a vector, and must be subject to the same laws as are vectors, may be seen at once from the form,

$$\nabla = i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z}.$$

Hence we can not write $p\nabla q = \nabla q . p$, when p is a quaternion, nor when p and q are both quaternions. If $f\rho$ be a vector function, we can not always place the operand immediately to the right of the operator, and must use subscripts.

Any vector function may be expressed,

$\sigma = i\xi + j\eta + k\zeta$, then

$$\nabla\sigma = \left(i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z}\right)(i\xi + j\eta + k\zeta)$$

$$= -\left(\frac{\delta\xi}{\delta x} + \frac{\delta\eta}{\delta y} + \frac{\delta\zeta}{\delta z}\right) - i\left(\frac{\delta\eta}{\delta z} + \frac{\delta\zeta}{\delta y}\right)$$

$$-j\left(\frac{\delta\xi}{\delta z}+\frac{\delta\zeta}{\delta x}\right)-k\left(\frac{\delta\eta}{\delta x}+\frac{\delta\xi}{\delta y}\right) \quad (11).$$

The operation ∇ may be expressed by a quaternion expression involving only symbols of partial differentiation. If we let $d\rho$, $d\rho'$, and $d\rho''$ represent any three non-coplanar differentials of ρ, then d, d', and d'' may be taken as the corresponding symbols of partial differentiation. Then from,

$$\nabla = i\frac{\delta}{\delta x}+j\frac{\delta}{\delta y}+k\frac{\delta}{\delta z}$$

$$\nabla = \frac{\delta x\,\delta y\,\delta z\left(i\frac{\delta}{\delta x}+j\frac{\delta}{\delta y}+k\frac{\delta}{\delta z}\right)}{\delta x\,\delta y\,\delta z}$$

$$= \frac{V.i\,\delta y\,\delta z\,\delta x\frac{\delta}{\delta x}+V.j\,\delta z\,\delta x\,\delta y\frac{\delta}{\delta y}+V.k\,\delta x\,\delta y\,\delta z\frac{\delta}{\delta z}}{S\,i\,\delta x\,j\,\delta y\,k\,\delta z}$$

since the quantities in the numerator are pure scalars, except i, j, k, and their product is also a scalar. Take

$$\rho = ix+jy+kz.$$

Then

$$d_t\rho = i\,dx+j\,dy+k\,dz$$

or, $d\rho = i\,\delta x$.

$$d'\rho = j\,\delta y$$
$$d''\rho = k\,\delta z$$

Then, remembering that $ij = k$; $jk = i$, etc., we have,

$$\nabla = -\frac{V d'\rho\, d''\rho . d + V d''\rho\, d\rho . d' + V d\rho\, d'\rho . d''}{S\, d\rho\, d'\rho\, d''\rho} \qquad (12)$$

Suppose that α, β, γ, are any constant vectors, and x, y, z, any scalar functions of ρ, then $\alpha X + \beta Y + \gamma Z$ may be taken to represent any vector function σ, ~~then~~

$$d\sigma = d(\alpha X + \beta Y + \gamma Z)$$
$$= dX.\alpha + dY.\beta + dZ.\gamma$$
$$= -(S\,d\rho\nabla X)\alpha - (S\,d\rho\nabla Y)\beta - (S\,d\rho\nabla Z)\gamma.$$

Since X, Y, Z, are scalar functions, we have from (4)

$$d\sigma = -[S\,d\rho\nabla](X\alpha + Y\beta + Z\gamma)$$
$$= -(S\,d\rho\nabla\sigma). \qquad (13)$$

If we replace σ by a quaternion function of ρ instead of taking a pure vector function, q can at once be broken up into the sum of a scalar and a vector, and the same result will be obtained. From the distributive property,

$$dq = d.Sq + d.Vq.$$

$$d.Sq = -S d\rho\nabla.Sq$$

$$d.Vq = -S d\rho\nabla.Vq.$$

$$\therefore dq = -(S d\rho\nabla.Sq + S d\rho\nabla.Vq)$$

$$= -(S d\rho\nabla.q).$$

This general result having been obtained whether q represent a quaternion or one or the other of its degraded forms, a scalar, or a vector, we are enabled to write, in symbolic language,

$$d = -S d\rho\nabla, \qquad (14)$$

in which d is taken to represent total differentiation. Equation (14) may be easily seen to represent the same thing as (12) if the symbol of total differentiation be represented by the partials.

If in equation (12) ρ represent a vector from the origin, take the small parallelepiped whose small edges are $d\rho$, $d'\rho$, and $d''\rho$, and whose centre is at the extremity of ρ. The areas of the faces are $\pm V d'\rho d''\rho$, $\pm V d''\rho d\rho$, and $\pm V d\rho d'\rho$. The mean value of q over the face may be taken as its value at the centre of the face, i.e., $q + dq$, if q be its value at the centre of the parallelepiped. Hence, from (12) ∇q is equal to the sum of the products of directed elements of the surface into the corresponding values of q, divided by the volume included.

If now we take, instead of a parallelepiped, any small closed surface, around the extremity of ρ, we may conceive the region enclosed by the surface divided into an infinite number of parallelepipeds, for each of which

$$\nabla q . dv = \int d\nu . q. \qquad (15)$$

∇q referring to the centre of the small parallelepiped. Summing,

$$\Sigma \nabla q . dv = \Sigma \int d\nu \, q \qquad (16)$$

If the surface is continuous the

inner faces are each measured twice, in opposite directions and hence contribute nothing to the value of the integral. Therefore, in the limit, only the bounding surfaces of the enclosed volume remain, and we may write,

$$\lim \Sigma \nabla q . dv = \int \nabla q . dv = \int dv . q,$$

and if the region be taken very small,

$$\nabla q = \lim \frac{1}{v} \int dv\, q$$

The value of ∇q at any point is then, the limit of the integral of the outwardly directed elements of any small closed surface surrounding the point, multiplied into the corresponding quaternion q, and divided by the volume enclosed by the surface.

This result is capable of many physical interpretations. In the case of q a scalar, we may assume that it represents a hydrostatic pressure p. Then $-dv . p$ is the pressure, in direction and magnitude, on the directed element, and $-\int d\nu . p = -v \nabla p$ is the resultant pressure on the entire surface, which urges the element in the direction in which p diminishes most rapidly.

In the case of q a vector, σ, the integral has two parts, one a vector, and one a scalar. Since,

$$\nabla \sigma = v^{-1}(S+V)\int d\nu . \sigma$$
$$= v^{-1}\int S\, d\nu . \sigma + v^{-1}\int V d\nu\, \sigma,$$

S and V being distributive symbols, the scalar part depends only on the components of the vectors normal to the surface, and the vector part of the tangential components. For the scalar part we have,

$$S\nabla\sigma = \frac{1}{v}\int S\, d\nu . \sigma \qquad (17)$$

If σ be taken to represent the displacement of a point in a body capable of any deformation, at the extremity of the vector ρ, the integral then represents the sum of the inward components of displacement of the elements of the small surface, i.e., is the diminution in volume.

If σ represent the flux of a fluid, the value of the integral measures the rate at which the inflow exceeds the outflow, i.e., if σ is the velocity and c is the density, $S\nabla(c\sigma)$ is the rate of increase in density. Hence $S\nabla\sigma$ was called by Maxwell the "convergence" of the vector σ.

As to the vector part,

$$V.\nabla\sigma = \lim \frac{1}{v} \int V.d\sigma \qquad (18),$$

it can be seen to represent the tangential components of the force. It is, therefore, a rotational phenomenon. Hence Maxwell[1] called it the "curl" or "version" of σ at the point P. If $V\nabla q = 0$, then q degenerates into a scalar quantity and we have

1. "Electricity and Magnetism", vol. 1. Art. 25. p28.

no rotation. Hence each of its constituents vanishes, and the displacements must be in the direction of, and proportional to the normal vectors of the series of surfaces. Hence, in any distorted series of particles, there is no compression if $S\nabla\sigma = 0$, and no rotation if $V\nabla\sigma = 0$. If $\nabla\sigma = 0$, then both $S\nabla\sigma = 0$, and $V\nabla\sigma = 0$, simultaneously, and if $\sigma = \alpha =$ any constant vector, there is merely transference.

If we take the case $\sigma = e\nabla f\rho$; $\nabla\sigma = e\nabla^2 f\rho$, which is a scalar. Therefore, in this case there occurs only compression and translation, the components being proportional to the density of the distribution of matter that would give the potential $f\rho$.

If r represent this density

then,

$\nabla^2 f\rho = 4\pi r.$ (19).

This equation, written in its ordinary analytical form,

$$\frac{\delta^2 V}{\delta x^2}+\frac{\delta^2 V}{\delta y^2}+\frac{\delta^2 V}{\delta z^2}+4\pi\rho=0,$$

is known as Poisson's equation, and if $\rho = 0$, it becomes Laplace's equation in three variables,

$$\frac{\delta^2 V}{\delta x^2}+\frac{\delta^2 V}{\delta y^2}+\frac{\delta^2 V}{\delta z^2}=0. \qquad (20)$$

The quantity V is called the potential function. The name potential function was first applied to it by Green in his essay, "On the Application of Mathematical Analysis to Electricity". Its use was first introduced by Laplace in his calculation of the attraction of the earth. Maxwell[1]

1. "Electricity and Magnetism" vol 1. p 72.

defined the potential as the line integral of the force;

$$V = \int (X\,dx + Y\,dy + Z\,dz).$$

Applied to electricity in its general form, it enables us to determine the distribution of electricity when we know the potential at every point.

Hamilton showed that Poisson's equation could be expressed in the symbolic language of quaternions as $\nabla^2 V = 4\pi\rho$, and Laplace's equation as $\nabla^2 V = 0$.

The treatment of Laplace's equation is extremely difficult. Clerk Maxwell used the method of conjugate functions and conformal representation for the case in which,

$$\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2} = 0.$$

In the general case in which

z is not zero, he used Spherical Harmonics, which, indeed, were invented by Laplace for use in the integration of his celebrated equation.[1] A Solid Harmonic is defined[2] by Maxwell as a homogeneous function of x, y, z, such that it satisfies Laplace's equation. The use of the operator ∇ greatly simplifies this problem.

A displacement in a space uniformly filled with points and bounded by a closed surface causes an increase or decrease in the number of points in the region. This excess can be measured in either of two ways. (1) By taking account of the increase of density at all points within the region.

1. "Mécanique Céleste". Book III
2. "Electricity and Magnetism". vol. 1. p. 165.

(2). By estimating the excess of those which pass inward over those which pass outward. Hence, in analytical language, for (1) we have,

$$\iiint\left(\frac{\delta^2 V}{\delta x^2}+\frac{\delta^2 V}{\delta y^2}+\frac{\delta^2 V}{\delta z^2}\right)d\tau.$$

The function within the parenthesis is was called[1] by Webster, after Lamé, "The Second Differential Parameter". For (2),

$$-\iint\frac{\delta V}{\delta n}\,dS.$$

These must be equal, being expressions for the same thing, then

$$\iiint\left(\frac{\delta^2 V}{\delta x^2}+\frac{\delta^2 V}{\delta y^2}+\frac{\delta^2 V}{\delta z^2}\right)d\tau=-\iint\frac{\delta V}{\delta n}\,dS.$$

In quaternion notation, this may be expressed,

$$\iiint S\nabla\sigma\, ds=\iint S.\sigma U\nu\, ds \qquad (21).$$

If σ represent the vector force,

1. "Theory of Electricity and Magnetism" p.64.

on a unit particle at the extremity of ρ, by any distribution, of matter, of electricity, or of magnetism, and if V be the potential, and r the corresponding density,

$$\sigma = \nabla V, \text{ and} \qquad (22)$$

$$\nabla \sigma = \nabla^2 V = 4\pi r$$

Then, since from (21)

$$\iiint S.\nabla \sigma ds = \iint S.\sigma U\nu ds,$$

$$4\pi \iiint r ds = \iint S\nabla V U\nu ds = 4\pi M. \qquad (23).$$

This is expressed analytically by Webster,[1]

$$4\pi M = \iint_s \frac{\delta V}{\delta n} dS = \iiint_\kappa \rho\, d\tau, \qquad (24)$$

or in the notation of Maxwell,[2]

$$\iint R \cos\varepsilon\, ds = \iiint\left(\frac{\delta X}{\delta x} + \frac{\delta Y}{\delta y} + \frac{\delta Z}{\delta z}\right) dx dy dz. \qquad (25)$$

If V and V_1 be any scalar functions of ρ, we can find the distribution of matter, etc., requis-

1. "Electricity and Magnetism" p. 151.
2. "Electricity and Magnetism" ~~vol~~. 1. p77.

ite to make either of them the potential at p. Let the necessary densities be r and r_1, respectively. Then,

$$\left.\begin{aligned} \nabla^2 V &= 4\pi r \\ \nabla^2 V_1 &= 4\pi r_1 \end{aligned}\right\} \text{ by Poisson's equation.}$$

$$\nabla(V\nabla V_1) = \nabla V . \nabla V_1 + V\nabla^2 V_1 \qquad (26)$$

Hence, in formula (23) if we put $\sigma = V\nabla V_1$, we obtain,

$$\iiint S.\nabla(V\nabla V_1)\,ds = -\iiint V\nabla^2 V_1\,ds + \iint V\,S.\nabla V_1 U\nu\,ds.$$
$$= -\iiint V_1\nabla^2 V\,ds + \iint V_1\,S.\nabla V U\nu\,ds. \quad (27).$$

This is the quaternion form for Green's Theorem, as may be readily seen by comparing with Maxwell[1], or with Webster[2], or Tait[3]. It was first given by Green in an essay, "On the Application of Mathematics to Electricity and Magnetism".

1. "Electricity and Magnetism" vol. 1. p. 108.
2. "Theory of Electricity and Magnetism" p. 63.
3. "On Green's and Other Allied Theorems." T. R. S. E. 1869-70.

Suppose V and V_1 equal, then,

$$\iiint(\nabla V)^2 ds = \iint V S.\nabla V U\nu ds. \qquad (28)$$

This shows that if ∇V is zero all over the surface, then it is zero all over the interior. If the surface is an equipotential one, then,

$$\iiint \nabla^2 V ds = V\iint S.\nabla V U\nu ds = V\iiint \nabla^2 ds,$$

from equation (21).

This shows that the potential within a closed surface is always constant. It is necessary, before we extend this theorem to vector functions, to assign a definite meaning to ∇^{-1}, an inverse operation to that denoted by ∇. The ~~equation~~ of Poisson, $\nabla^2\sigma = \tau$, expresses that the constituents of σ are the potentials of certain distributions of matter, electricity or magnetism, whose densities are the constituents of τ. Then we define ∇^{-1} by the equation

$$\nabla\sigma = \nabla^{-1}\tau, \qquad (29)$$

By comparing with Webster, we see that $\nabla^{-1}\tau$ is proportional to what he calls the operator Pot., and the proportionality factor is $\frac{1}{4\pi}$ i.e.,
$$\nabla^{-1}\tau = \frac{1}{4\pi} \mathrm{Pot}\,\tau \iiint_{\infty} \frac{\tau}{r}\, d\tau,$$
Webster's notation being changed to avoid ambiguity.

If σ represent any vector whatever, $\sigma = iV + jV_1 + kV_2$, we may write (21) in the form,
$$\iiint \nabla^2 \sigma\, ds = \iint S(U_\nu \nabla)\sigma\, ds.$$
If $\nabla^2\sigma = \tau$, this becomes,
$$\iiint \tau\, ds = \iint S.(U_\nu \nabla^{-1})\tau\, ds, \qquad (30)$$
which gives us a method of representing by a surface integral a vector space integral.

The line integrals with which we have to deal in mathematical physics are integrals of vector point functions. If σ represent the vector function, then the line integral may be expressed as

$\int S.\sigma d\tau$; where τ is the vector of any point in the small closed curve drawn from a point within it, and in its plane. Then $\sigma = \sigma_0 - S(\tau\nabla)\sigma_0$ where σ_0 is that value of σ at the origin of τ, and therefore,

$$\int S\sigma d\tau = \int S(\sigma_0 - S(\tau\nabla)\sigma_0)d\tau.$$

Since the integral is taken around a closed curve, its value is zero, and

$$\int S.\tau\nabla.S.\sigma_0 d\tau = \tfrac{1}{2}S.\nabla(\tau S\sigma_0\tau - \sigma_0\int V\tau d\tau).$$

$$\therefore \tfrac{1}{2}\int V\tau d\tau = ds\,U\nu,$$

where ds is the small elemental area, and $U\nu$ is a vector perpendicular to its plane. Hence,

$$\int S.\sigma_0 d\tau = S.\nabla\sigma_0 U\nu ds,$$

And, if we consider any surface as broken up into a number of such elements, we have, for a finite unclosed surface,

$$\int S.\sigma ds = \iint \sigma U\nu ds. \qquad (31).$$

This result is the quaternion

expression for what is known as "Stokes Theorem", as may be seen by comparing Webster[1], who gives it in the analytical form,

$$\int(X\,dx + Y\,dy + Z\,dz) = \int R\,ds$$
$$= \iint w \cos w.n\,dS.$$
$$= \iint(\xi \cos(nx) + \eta \cos(ny) + \zeta \cos(nz))\,ds. \quad (32).$$

in which,

$$\xi = \frac{\delta Z}{\delta y} - \frac{\delta Y}{\delta z};$$

$$\eta = \frac{\delta X}{\delta z} - \frac{\delta Z}{\delta x};$$

$$\zeta = \frac{\delta Y}{\delta x} - \frac{\delta X}{\delta y}.$$

The quantity w defined by the above differential equation, (32), is related to R, the vector point function. It is called[1] by Maxwell the "curl" of R. It may be expressed by the use of the operator ∇, thus;

1. "Electricity and Magnetism" p. 28. (vol. 1.)

$$\nabla R = \left(i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z}\right)(iX + jY + kZ)$$

$$= -\left(\frac{\delta X}{\delta x} + \frac{\delta Y}{\delta y} + \frac{\delta Z}{\delta z}\right) + i\left(\frac{\delta Z}{\delta y} - \frac{\delta Y}{\delta z}\right)$$

$$+ j\left(\frac{\delta X}{\delta z} - \frac{\delta Z}{\delta x}\right) + k\left(\frac{\delta Y}{\delta x} - \frac{\delta X}{\delta y}\right).$$

$$\therefore \text{curl } R = V\nabla R = i\xi + j\eta + k\varsigma = \omega. \quad (33)$$

If σ be taken to represent the direction and intensity of the magnetization per unit volume at element ds, and the magnet be placed in a magnetic field whose potential is u, we have, if E represent potential energy, by (21)

$$E = -\iiint S.\sigma\nabla u\, ds$$

$$= \iiint u\, S.\nabla\sigma\, ds - \iint u\, S.\sigma U\nu\, ds. \quad (34)$$

The first integral considers the volume density, $S.\nabla\sigma$, the second, the surface density, $-S\sigma U\nu$. The quantity $-S\nabla u$ is called by Webster,[1] the

1. "Theory of Electricity and Magnetism" p.66,

"divergence" of u. The condition, $S\nabla u = 0$ is termed the solenoidal condition. If $V\nabla\sigma = 0$, we have what is called, after Thomson[2], a lamellar distribution. If $V\nabla(u\sigma) = 0$, where u is a scalar multiplier, or $S.\sigma.\nabla\sigma = 0$, we have the condition[3] for a complex lamellar distribution.

Such examples of the uses of the operator ∇ might be multiplied almost indefinitely, but it is beyond the scope of this article to attempt to give a full treatment of the subject.

1. "Theory of Electricity and Magnetism" p.21.
2. "Philosophical Transactions" 1852
3. Webster. "Theory of Elec. and Mag." p.68.

Bibliography.

I. "Lectures on Quaternions" W. R. Hamilton.

II. "Elements of Quaternions" W. R. Hamilton. Second Edition. Edited by C. J. Joly. vol. II

III. "Quaternions" Second Edition. P. G. Tait.

IV. "Theory of Electricity and Magnetism" A. G. Webster.

V. "Treatise on Natural Philosophy" Thomson and Tait.

VI. "A Treatise on Electricity and Magnetism" vol. I. James Clerk Maxwell.

VII. "A Treatise on the Analytical Geometry of Three Dimensions" 3rd edition George Salmon.

VIII. "The Functions of Laplace, Lamé, and Bessel" Todhunter.

IX. "Traité des Fonctions Inverses" M. Lamé.

X. "On Green's and Other Allied Theorems" Tait. Trans. R. S. E. 1869-'70.

XI. "On Some Quaternion Integrals" Tait. Proceedings. R. S. E. 1870-'71